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Research Article

On k-Quasiclass A Operators

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An operator $T \in B(\mathscr{H})$ is called k-quasiclass A if $T^{*k}(|T^2| - |T|^2)T^k \geq 0$ for a positive integer k, which is a common generalization of quasiclass A. In this paper, firstly we prove some inequalities of this class of operators; secondly we prove that if T is a k-quasiclass A operator, then T is isoloid and $T - \lambda$ has finite ascent for all complex number λ; at last we consider the tensor product for k-quasiclass A operators.

1. Introduction

Throughout this paper let $\mathscr{H}$ be a separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $B(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$.

Let $T \in B(\mathscr{H})$ and let λ_0 be an isolated point of $\sigma(T)$. Here $\sigma(T)$ denotes the spectrum of T. Then there exists a small enough positive number $r > 0$ such that

$$\{\lambda \in C : |\lambda - \lambda_0| \leq r\} \cap \sigma(T) = \{\lambda_0\}. \tag{1.1}$$

Let

$$E = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = r} (\lambda - T)^{-1} d\lambda. \tag{1.2}$$

E is called the Riesz idempotent with respect to λ_0, and it is well known that E satisfies $E^2 = E$, $TE = ET$, $\sigma(T|_{E\mathscr{H}}) = \{\lambda_0\}$, and $\ker((T - \lambda_0)^n) \subset E\mathscr{H}$ for all positive integers n. Stampfli [1] proved that if T is hyponormal (i.e., operators such that $T^*T - TT^* \geq 0$), then

$$E \text{ is self-adjoint and } E\mathscr{H} = \ker(T - \lambda_0) = \ker\big((T - \lambda_0)^*\big). \tag{1.3}$$

After that many authors extended this result to many other classes of operators. Chō and Tanahashi [2] proved that (1.3) holds if T is either p-hyponormal or log-hyponormal. In the case $\lambda_0 \neq 0$, the result was further shown by Tanahashi and Uchiyama [3] to hold for p-quasihyponormal operators, by Tanahashi et al. [4] to hold for (p,k)-quasihyponormal operators and by Uchiyama and Tanahashi [5] and Uchiyama [6] for class A and paranormal operators. Here an operator T is called p-hyponormal for $0 < p \leq 1$ if $(T^*T)^p - (TT^*)^p \geq 0$, and log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. An operator T is called (p,k) -quasihyponormal if $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$, where $0 < p \leq 1$ and k is a positive integer; especially, when $p = 1$, $k = 1$, and $p = k = 1$, T is called k-quasihyponormal, p-quasihyponormal, and quasihyponormal, respectively. And an operator T is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathscr{H}$; normaloid if $\|T^n\| = \|T\|^n$ for all positive integers n. p-hyponormal, log-hyponormal, p-quasihyponormal, (p,k)-quasihyponormal, and paranormal operators were introduced by Aluthge [7], Tanahashi [8], S. C. Arora and P. Arora [9], Kim [10], and Furuta [11, 12], respectively.

In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators, Furuta et al. [13] introduced a very interesting class of bounded linear Hilbert space operators: class A defined by $|T^2| - |T|^2 \geq 0$, where $|T| = (T^*T)^{1/2}$ which is called the absolute value of T and they showed that class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators. Class A operators have been studied by many researchers, for example, [5, 14–19].

Recently Jeon and Kim [20] introduced quasiclass A (i.e., $T^*(|T^2| - |T|^2)T \geq 0$) operators as an extension of the notion of class A operators, and they also proved that (1.3) holds for this class of operators when $\lambda_0 \neq 0$. It is interesting to study whether Stampli's result holds for other larger classes of operators.

In [21], Tanahashi et al. considered an extension of quasi-class A operators, similar in spirit to the extension of the notion of p-quasihyponormality to (p,k) -quasihyponormality, and prove that (1.3) holds for this class of operators in the case $\lambda_0 \neq 0$.

Definition 1.1. $T \in B(\mathscr{H})$ is called a k-quasiclass A operator for a positive integer k if

$$T^{*k}\left(\left|T^2\right| - |T|^2\right)T^k \geq 0. \tag{1.4}$$

Remark 1.2. In [21], this class of operators is called quasi-class (A, k).

It is clear that the class of quasi-class A operators $\subseteq$ the class of k-quasiclass A operators and

$$\text{the class of } k\text{-quasiclass A operators} \subseteq \text{the class of } (k+1)\text{-quasiclass A operators}. \tag{1.5}$$

We show that the inclusion relation (1.5) is strict, by an example which appeared in [20].

Example 1.3. Given a bounded sequence of positive numbers $\{\alpha_i\}_{i=0}^{\infty}$, let T be the unilateral weighted shift operator on l^2 with the canonical orthonormal basis $\{e_n\}_{n=0}^{\infty}$ by $Te_n = \alpha_n e_{n+1}$ for all $n \geq 0$, that is,

$$T = \begin{pmatrix} 0 & & & & \\ \alpha_0 & 0 & & & \\ & \alpha_1 & 0 & & \\ & & \alpha_2 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}. \tag{1.6}$$

Straightforward calculations show that T is a k-quasiclass A operator if and only if $\alpha_k \leq \alpha_{k+1} \leq \alpha_{k+2} \leq \cdots$. So if $\alpha_{k+1} \leq \alpha_{k+2} \leq \alpha_{k+3} \leq \cdots$ and $\alpha_k > \alpha_{k+1}$, then T is a $(k+1)$-quasiclass A operator, but not a k-quasiclass A operator.

In this paper, firstly we consider some inequalities of k-quasiclass A operators; secondly we prove that if T is a k-quasiclass A operator, then T is isoloid and $T - \lambda$ has finite ascent for all complex number λ; at last we give a necessary and sufficient condition for $T \otimes S$ to be a k-quasiclass A operator when T and S are both non-zero operators.

2. Results

In the following lemma, Tanahashi, Jeon, Kim, and Uchiyama studied the matrix representation of a k-quasiclass A operator with respect to the direct sum of $\overline{\operatorname{ran}(T^k)}$ and its orthogonal complement.

Lemma 2.1 (see [21]). *Let $T \in B(\mathscr{H})$ be a k-quasiclass A operator for a positive integer k and let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathscr{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker T^{*k}$ be 2×2 matrix expression. Assume that $\operatorname{ran} T^k$ is not dense, then T_1 is a class A operator on $\overline{\operatorname{ran}(T^k)}$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. Consider the matrix representation of T with respect to the decomposition $\mathscr{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker T^{*k}$: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$. Let P be the orthogonal projection of $\mathscr{H}$ onto $\overline{\operatorname{ran}(T^k)}$. Then $T_1 = TP = PTP$. Since T is a k-quasiclass A operator, we have

$$P\left(\left|T^2\right| - |T|^2\right)P \geq 0. \tag{2.1}$$

Then

$$\left|T_1^2\right| = \left(PT^*PT^*TPTP\right)^{1/2} = \left(PT^*T^*TTP\right)^{1/2} = \left(P\left|T^2\right|^2 P\right)^{1/2} \geq P\left|T^2\right|P \tag{2.2}$$

by Hansen's inequality [22]. On the other hand

$$|T_1|^2 = T_1^* T_1 = PT^*TP = P|T|^2 P \leq P\left|T^2\right|P. \tag{2.3}$$

Hence

$$\left|T_1^2\right| \geq |T_1|^2. \tag{2.4}$$

That is, T_1 is a class A operator on $\overline{\operatorname{ran}(T^k)}$.

For any $x = (x_1, x_2) \in \mathscr{H}$,

$$\left\langle T_3^k x_2, x_2 \right\rangle = \left\langle T^k (I-P)x, (I-P)x \right\rangle = \left\langle (I-P)x, T^{*k}(I-P)x \right\rangle = 0, \tag{2.5}$$

which implies $T_3^k = 0$.

Since $\sigma(T) \cup \mathfrak{G} = \sigma(T_1) \cup \sigma(T_3)$, where $\mathfrak{G}$ is the union of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_3)$ by [23, Corollary 7], and $\sigma(T_3) = 0$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$. □

Theorem 2.2. *Let $T \in B(\mathscr{H})$ be a k-quasiclass A operator for a positive integer k. Then the following assertions hold.*

(1) $\|T^{n+2}x\|\|T^n x\| \geq \|T^{n+1}x\|^2$ *for all $x \in \mathscr{H}$ and all positive integers $n \geq k$.*

(2) *If $T^n = 0$ for some positive integer $n \geq k$, then $T^{k+1} = 0$.*

(3) $\|T^{n+1}\| \leq \|T^n\| r(T)$ *for all positive integers $n \geq k$, where $r(T)$ denotes the spectral radius of T.*

To give a proof of Theorem 2.2, the following famous inequality is needful.

Lemma 2.3 (Hölder-McCarthy's inequality [24]). *Let $A \geq 0$. Then the following assertions hold.*

(1) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r \|x\|^{2(1-r)}$ *for $r > 1$ and all $x \in \mathscr{H}$.*

(2) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r \|x\|^{2(1-r)}$ *for $r \in [0,1]$ and all $x \in \mathscr{H}$.*

Proof of Theorem 2.2. (1) Since it is clear that k-quasiclass A operators are $(k+1)$-quasiclass A operators, we only need to prove the case $n = k$. Since

$$\begin{aligned} \langle T^{*k}|T|^2 T^k x, x\rangle &= \langle T^{*k}T^*TT^k x, x\rangle = \left\|T^{k+1}x\right\|^2, \\ \left\langle T^{*k}\left|T^2\right|T^k x, x\right\rangle &= \left\langle \left|T^2\right|T^k x, T^k x\right\rangle \\ &\leq \left\langle T^*T^*TTT^k x, T^k x\right\rangle^{1/2} \left\|T^k x\right\|^{2(1-1/2)} \\ &= \left\|T^{k+2}x\right\|\left\|T^k x\right\| \end{aligned} \tag{2.6}$$

by Hölder-McCarthy's inequality, we have

$$\left\|T^{k+2}x\right\|\left\|T^k x\right\| \geq \left\|T^{k+1}x\right\|^2 \tag{2.7}$$

for T is a k-quasiclass A operator.

(2) If $n = k, k+1$, it is obvious that $T^{k+1} = 0$. If $T^{k+2} = 0$, then $T^{k+1} = 0$ by (1). The rest of the proof is similar.

(3) We only need to prove the case $n = k$, that is,

$$\left\|T^{k+1}\right\| \leq \left\|T^{k}\right\| r(T). \tag{2.8}$$

If $T^n = 0$ for some $n \geq k$, then $T^{k+1} = 0$ by (2) and in this case $r(T) = (r(T^{k+1}))^{1/(k+1)} = 0$. Hence (3) is clear. Therefore we may assume $T^n \neq 0$ for all $n \geq k$. Then

$$\frac{\|T^{k+1}\|}{\|T^{k}\|} \leq \frac{\|T^{k+2}\|}{\|T^{k+1}\|} \leq \frac{\|T^{k+3}\|}{\|T^{k+2}\|} \leq \cdots \leq \frac{\|T^{mk}\|}{\|T^{mk-1}\|} \tag{2.9}$$

by (1), and we have

$$\left(\frac{\|T^{k+1}\|}{\|T^{k}\|}\right)^{mk-k} \leq \frac{\|T^{k+1}\|}{\|T^{k}\|} \times \frac{\|T^{k+2}\|}{\|T^{k+1}\|} \times \cdots \times \frac{\|T^{mk}\|}{\|T^{mk-1}\|} = \frac{\|T^{mk}\|}{\|T^{k}\|}. \tag{2.10}$$

Hence

$$\left(\frac{\|T^{k+1}\|}{\|T^{k}\|}\right)^{k-(k/m)} \leq \frac{\|T^{mk}\|^{1/m}}{\|T^{k}\|^{1/m}}. \tag{2.11}$$

By letting $m \to \infty$, we have

$$\left\|T^{k+1}\right\|^{k} \leq \left\|T^{k}\right\|^{k} (r(T))^{k}, \tag{2.12}$$

that is,

$$\left\|T^{k+1}\right\| \leq \left\|T^{k}\right\| r(T). \tag{2.13}$$

□

Lemma 2.4 (see [21]). *Let $T \in B(\mathscr{H})$ be a k-quasiclass A operator for a positive integer k. If $\lambda \neq 0$ and $(T-\lambda)x = 0$ for some $x \in \mathscr{H}$, then $(T-\lambda)^* x = 0$.*

Proof. We may assume that $x \neq 0$. Let $\mathcal{M}_0$ be a span of $\{x\}$. Then $\mathcal{M}_0$ is an invariant subspace of T and

$$T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \mathscr{H} = \mathcal{M}_0 \oplus \mathcal{M}_0^{\perp}. \tag{2.14}$$

Let P be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}_0$. It suffices to show that $T_2 = 0$ in (2.14). Since T is a k-quasiclass A operator, and $x = T^k(x/\lambda^k) \in \overline{\operatorname{ran}(T^k)}$, we have

$$P\left(\left|T^2\right| - |T|^2\right)P \geq 0. \tag{2.15}$$

We remark

$$P\left|T^2\right|^2 P = PT^*T^*TTP = PT^*PT^*TPTP = \begin{pmatrix} |\lambda|^4 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.16}$$

Then by Hansen's inequality and (2.15), we have

$$\begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} = \left(P\left|T^2\right|^2 P\right)^{1/2} \geq P\left|T^2\right|P \geq P|T|^2P = PT^*TP = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.17}$$

Hence we may write

$$\left|T^2\right| = \begin{pmatrix} |\lambda|^2 & A \\ A^* & B \end{pmatrix}. \tag{2.18}$$

We have

$$\begin{aligned} \begin{pmatrix} |\lambda|^4 & 0 \\ 0 & 0 \end{pmatrix} &= P\left|T^2\right|\left|T^2\right|P \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |\lambda|^2 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} |\lambda|^2 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} |\lambda|^4 + AA^* & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.19}$$

This implies $A = 0$ and $|T^2|^2 = \begin{pmatrix} |\lambda|^4 & 0 \\ 0 & B^2 \end{pmatrix}$. On the other hand,

$$\begin{aligned} \left|T^2\right|^2 &= T^*T^*TT \\ &= \begin{pmatrix} \overline{\lambda} & 0 \\ T_2^* & T_3^* \end{pmatrix} \begin{pmatrix} \overline{\lambda} & 0 \\ T_2^* & T_3^* \end{pmatrix} \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix} \\ &= \begin{pmatrix} |\lambda|^4 & \overline{\lambda}^2(\lambda T_2 + T_2T_3) \\ \lambda^2(\lambda T_2 + T_2T_3)^* & |\lambda T_2 + T_2T_3|^2 + \left|T_3^2\right|^2 \end{pmatrix}. \end{aligned} \tag{2.20}$$

Hence $\lambda T_2 + T_2T_3 = 0$ and $B = |T_3^2|$. Since T is a k-quasiclass A operator, by a simple calculation we have

$$0 \leq T^{*k}\left(\left|T^2\right| - |T|^2\right)T^k$$
$$= \begin{pmatrix} 0 & (-1)^{k+1}\overline{\lambda}|\lambda|^{2k}T_2 \\ (-1)^{k+1}\lambda|\lambda|^{2k}T_2^* & (-1)^{k+1}|\lambda|^{2k}|T_2|^2 + T_3^{*k}\left|T_3^2\right|T_3^k - \left|T_3^{k+1}\right|^2 \end{pmatrix}. \tag{2.21}$$

Recall that $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X, Z \geq 0$ and $Y = X^{1/2}WZ^{1/2}$ for some contraction W. Thus we have $T_2 = 0$. This completes the proof. □

Lemma 2.5 (see [25]). *If T satisfies $\ker(T-\lambda) \subseteq \ker(T-\lambda)^*$ for some complex number λ, then $\ker(T-\lambda) = \ker(T-\lambda)^n$ for any positive integer n.*

Proof. It suffices to show $\ker(T-\lambda) = \ker(T-\lambda)^2$ by induction. We only need to show $\ker(T-\lambda)^2 \subseteq \ker(T-\lambda)$ since $\ker(T-\lambda) \subseteq \ker(T-\lambda)^2$ is clear. In fact, if $(T-\lambda)^2x = 0$, then we have $(T-\lambda)^*(T-\lambda)x = 0$ by hypothesis. So we have $\|(T-\lambda)x\|^2 = \langle (T-\lambda)^*(T-\lambda)x, x\rangle = 0$, that is, $(T-\lambda)x = 0$. Hence $\ker(T-\lambda)^2 \subseteq \ker(T-\lambda)$. □

An operator is said to have finite ascent if $\ker T^n = \ker T^{n+1}$ for some positive integer n.

Theorem 2.6. *Let $T \in B(\mathscr{H})$ be a k-quasiclass A operator for a positive integer k. Then $T-\lambda$ has finite ascent for all complex number λ.*

Proof. We only need to show the case $\lambda = 0$ because the case $\lambda \neq 0$ holds by Lemmas 2.4 and 2.5.

In the case $\lambda = 0$, we shall show that $\ker T^{k+1} = \ker T^{k+2}$. It suffices to show that $\ker T^{k+2} \subseteq \ker T^{k+1}$ since $\ker T^{k+1} \subseteq \ker T^{k+2}$ is clear. Now assume that $T^{k+2}x = 0$. We may assume $T^kx \neq 0$ since if $T^kx = 0$, it is obvious that $T^{k+1}x = 0$. By Hölder-McCarthy's inequality, we have

$$\begin{aligned} 0 = \left\|T^{k+2}x\right\| &= \left\langle T^{k+2}x, T^{k+2}x\right\rangle^{1/2} \\ &= \left\langle \left|T^2\right|^2 T^kx, T^kx\right\rangle^{1/2} \\ &\geq \left\langle \left|T^2\right| T^kx, T^kx\right\rangle \left\|T^kx\right\|^{-1} \\ &\geq \left\langle |T|^2 T^kx, T^kx\right\rangle \left\|T^kx\right\|^{-1} \\ &= \left\|T^{k+1}x\right\|^2 \left\|T^kx\right\|^{-1}. \end{aligned} \tag{2.22}$$

So we have $T^{k+1}x = 0$, which implies $\ker T^{k+2} \subseteq \ker T^{k+1}$. Therefore $\ker T^{k+1} = \ker T^{k+2}$. □

In the following lemma, Tanahashi, Jeon, Kim, and Uchiyama extended the result (1.3) to k-quasiclass A operators in the case $\lambda_0 \neq 0$.

Lemma 2.7 (see [21]). *Let $T \in B(\mathcal{H})$ be a k-quasiclass A operator for a positive integer k. Let λ_0 be an isolated point of $\sigma(T)$ and E the Riesz idempotent for λ_0. Then the following assertions hold.*

(1) *If $\lambda_0 \neq 0$, then E is self-adjoint and*

$$E\mathcal{H} = \ker(T - \lambda_0) = \ker\left((T - \lambda_0)^*\right). \tag{2.23}$$

(2) *If $\lambda_0 = 0$, then $E\mathcal{H} = \ker(T^{k+1})$.*

An operator T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T.

Theorem 2.8. *Let $T \in B(\mathcal{H})$ be a k-quasiclass A operator for a positive integer k. Then T is isoloid.*

Proof. Let $\lambda \in \sigma(T)$ be an isolated point. If $\lambda \neq 0$, by (1) of Lemma 2.7, $\ker(T-\lambda) = E\mathcal{H} \neq \{0\}$ for $E \neq 0$. Therefore λ is an eigenvalue of T. If $\lambda = 0$, by (2) of Lemma 2.7, $\ker(T^{k+1}) = E\mathcal{H} \neq \{0\}$ for $E \neq 0$. So we have $\ker(T) \neq \{0\}$. Therefore 0 is an eigenvalue of T. This completes the proof. □

Let $T \otimes S$ denote the tensor product on the product space $\mathcal{H} \otimes \mathcal{H}$ for nonzero $T, S \in B(\mathcal{H})$. The following theorem gives a necessary and sufficient condition for $T \otimes S$ to be a k-quasiclass A operator, which is an extension of [20, Theorem 4.2].

Theorem 2.9. *Let $T, S \in B(\mathcal{H})$ be nonzero operators. Then $T \otimes S$ is a k-quasiclass A operator if and only if one of the following assertions holds*

(1) *$T^{k+1} = 0$ or $S^{k+1} = 0$.*

(2) *T and S are k-quasiclass A operators.*

Proof. It is clear that $T \otimes S$ is a k-quasiclass A operator if and only if

$$\begin{aligned}
&(T \otimes S)^{*k}\left(\left|(T \otimes S)^2\right| - |T \otimes S|^2\right)\ (T \otimes S)^k \geq 0 \\
&\Longleftrightarrow T^{*k}\left(\left|T^2\right| - |T|^2\right)T^k \otimes S^{*k}\left|S^2\right|S^k + T^{*k}|T|^2T^k \otimes S^{*k}\left(\left|S^2\right| - |S|^2\right)S^k \geq 0 \\
&\Longleftrightarrow T^{*k}\left|T^2\right|T^k \otimes S^{*k}\left(\left|S^2\right| - |S|^2\right)S^k + T^{*k}\left(\left|T^2\right| - |T|^2\right)T^k \otimes S^{*k}|S|^2S^k \geq 0.
\end{aligned} \tag{2.24}$$

Therefore the sufficiency is clear.

To prove the necessary, suppose that $T \otimes S$ is a k-quasiclass A operator. Let $x, y \in \mathcal{H}$ be arbitrary. Then we have

$$\left\langle T^{*k}\left(\left|T^2\right| - |T|^2\right)T^k x, x\right\rangle \langle S^{*k}\left|S^2\right|S^k y, y\rangle + \left\langle T^{*k}|T|^2T^k x, x\right\rangle \left\langle S^{*k}\left(\left|S^2\right| - |S|^2\right)S^k y, y\right\rangle \geq 0. \tag{2.25}$$

It suffices to prove that if (1) does not hold, then (2) holds. Suppose that $T^{k+1} \neq 0$ and $S^{k+1} \neq 0$. To the contrary, assume that T is not a k-quasiclass A operator, then there exists $x_0 \in \mathscr{H}$ such that

$$\left\langle T^{*k}\left(\left|T^2\right| - |T|^2\right)T^k x_0, x_0\right\rangle = \alpha < 0, \qquad \left\langle T^{*k}|T|^2 T^k x_0, x_0\right\rangle = \beta > 0. \tag{2.26}$$

From (2.25) we have

$$\alpha\left\langle S^{*k}\left|S^2\right|S^k y, y\right\rangle + \beta\left\langle S^{*k}\left(\left|S^2\right| - |S|^2\right)S^k y, y\right\rangle \geq 0 \quad \forall y \in \mathscr{H}, \tag{2.27}$$

that is,

$$(\alpha + \beta)\left\langle S^{*k}\left|S^2\right|S^k y, y\right\rangle \geq \beta\left\langle S^{*k}|S|^2 S^k y, y\right\rangle \tag{2.28}$$

for all $y \in \mathscr{H}$. Therefore S is a k-quasiclass A operator. As the proof in Theorem 2.2 (1), we have

$$\left\langle S^{*k}|S|^2 S^k y, y\right\rangle = \left\|S^{k+1}y\right\|^2, \qquad \left\langle S^{*k}\left|S^2\right|S^k y, y\right\rangle \leq \left\|S^{k+2}y\right\|\left\|S^k y\right\|. \tag{2.29}$$

So we have

$$(\alpha + \beta)\left\|S^{k+2}y\right\|\left\|S^k y\right\| \geq \beta\left\|S^{k+1}y\right\|^2 \tag{2.30}$$

for all $y \in \mathscr{H}$ by (2.28). Because S is a k-quasiclass A operator, from Lemma 2.1 we can write $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ on $\mathscr{H} = \overline{\operatorname{ran}(S^k)} \oplus \ker S^{*k}$, where S_1 is a class A operator (hence it is normaloid). By (2.30) we have

$$(\alpha + \beta)\left\|S_1^2\eta\right\|\|\eta\| \geq \beta\|S_1\eta\|^2 \quad \forall \eta \in \overline{\operatorname{ran}(S^k)}. \tag{2.31}$$

So we have

$$(\alpha + \beta)\|S_1\|^2 = (\alpha + \beta)\left\|S_1^2\right\| \geq \beta\|S_1\|^2, \tag{2.32}$$

where equality holds since S_1 is normaloid.

This implies that $S_1 = 0$. Since $S^{k+1}y = S_1 S^k y = 0$ for all $y \in \mathscr{H}$, we have $S^{k+1} = 0$. This contradicts the assumption $S^{k+1} \neq 0$. Hence T must be a k-quasiclass A operator. A similar argument shows that S is also a k-quasiclass A operator. The proof is complete. □

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